WIRE RESPONSE DUE TO A MOVING FORCE

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Abstract: High speed trains are characterized by high stability at high velocity and ride comfort, but all these features are conditioned by the problem of ensuring stable current collection that is supplied by the pantograph/catenary system. The catenary represents a complex structure consisting in a contact wire, messenger wire, droppers, supporting brackets, and registration arms. When the train velocity increases, the interaction between the pantograph and the catenary may be the cause of the contact loss and this aspect affects the train performances. The issue of the pantograph/catenary interaction is a part of the classical moving load problem, and two approaches could be considered, respectively considering or not the degrees of freedom of the pantograph. When the coupling between the vehicle and the catenary is overlooked, the interaction model consists of a moving force travelling on the flexible structure of the catenary. This article focuses on the catenary response to a mobile force by applying a method differing from the previous research where used the Fourier transform. This approach initiates in the properties of the Green function associated to the differential operator of the catenary model.

1. INTRODUCTION

The vibrations of the pantograph-catenary system have been constantly studied in the last 40 years once the travelling speed has much increased. While considering the catenary configuration to be seen as an infinite structure with a periodical variation of the elasticity, the vibrations of the pantograph-catenary system fall within the issue of the vibrations of a moving system on an elastic structure. This type of problems will trigger two perspectives that are essentially complementary and help to understand the phenomena of interaction between the moving and the elastic structure. The former perspective concerns solving the problem of the structure's response to a moving load, which means that the mobile action is reduced to the interaction force, not taking into account the degrees of freedom in the moving system [1, 2].

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The latter perspective regards the complete solving of the problem by considering the degrees of freedom of the mobile system. In a nutshell, the start is from the dynamic or static rigidity of the contact string so as to equate the catenary model with distributed parameters with one with concentrated and time variable parameters, depending on velocity and distance between the catenary supporters. This model is 'assembled' with the pantograph's, which is described by an oscillating system with one with up to three degrees of freedom [3, 4, 5].

Most of the studies concerning the catenary response to a moving force prefer its representation by linear models based on the hypotheses of the vibrant wire. Thus, the simplest model reduces the entire catenary structure to the contact string assimilated by an infinite wire connected to a fixed base by elastic elements and of uniformly distributed damping (Winkler foundation with a viscous damping). This is how the influence of the elasticity variation is neglected, due to the supporters and the hangers. The results derived from this model are basic and they can be used for a comparison, as did *A. Metrikine* [1] in order to highlight the non-linear effect brought about by the hangers.

2. THE MECHANICAL MODEL

Figure 1 shows a simple catenary system, which consists of the equidistant supporters (1) supporting the messenger wire (3). Each support has a cable arm (2) from that the contact wire is connected. The level of the contact wire position is corrected via the hangers (4). The path of the contact wire takes a zigzag shape, due to the steady arms. Both messenger and contact wires are stressed.
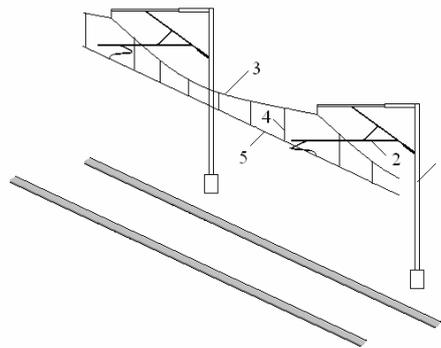

Fig. 1. Simple catenary system:
1. supporter; 2. steady arm; 3. messenger wire; 4. hanger; 5. contact wire.

Further, the simplest model of catenary will be considered, namely an infinite wire on a continuous elastic support (fig 2). The catenary wire has the transversal section *A* and is made up of a material with density ρ. At the same time, the wire is tensed by force *T*. The elastic support contains elastic and of damping elements with linear characteristics, uniformly distributed along the catenary.
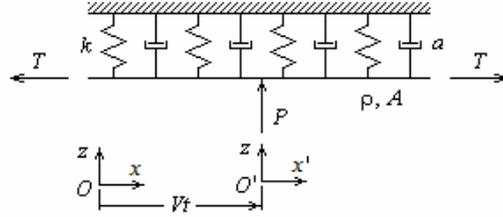
Fig. 2. Mechanical model of the catenary system.

A constant force P acts upon the wire and moves at a constant speed V. The wire movement reports to the fixed system Oxz. Also, a moving system attaches against the force $O'x'z$ with

$$x = Vt + x' \tag{1}$$

For a non-moving observer, the wire will start moving more and more as the force is getting closer to the observation point and will reach the maximum value when the force passes by the observer. The observer will perceive the wire movement as a wave travelling along it. Should the observer moves with the force P, the wire shape is stabilized on either side of the force at a certain time after travelling starts when the natural vibrations are damped, since the observer travels along with the wave derived from the moving force.

The wire movement equation reported to the fixed referential is

$$m\frac{\partial^2 w}{\partial t^2} + a\frac{\partial w}{\partial t} + kw - T\frac{\partial^2 w}{\partial x^2} = -mg + P\delta(x - Vt), \tag{2}$$

where $m = \rho A$ is the wire mass on the length unit, k – elastic constant, and a – the damping constant of the continuous elastic support.

Considering the steady state behaviour, the change of variable (1) is recommended, where the movement is reported to the moving referential. The equation writes as

$$(V^2 - c^2)\frac{\partial^2 w}{\partial x'^2} - 2\zeta\omega_0 V\frac{\partial w}{\partial x'} + 2V\frac{\partial^2 w}{\partial x'\partial t} + \omega_0^2 w + 2\zeta\omega_0\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial t^2} = -g + p\delta(x') \tag{3}$$

where ζ is the damping degree (only the case of the sub-critical damping is considered, $\zeta < 1$), ω_0 – the natural wire pulsation on the elastic support, c – the wave propagation speed through the wire and $p = P/m$. The calculation relations for the parameters are as follows

$$\zeta = \frac{a}{2\sqrt{mk}}, \quad \omega_0^2 = \frac{k}{m}, c = \sqrt{\frac{T}{m}}.$$

We are interested in the steady-state response and, due to that, the derivations in respect to the time are zero

$$(V^2 - c^2)\frac{\partial^2 w}{\partial x'^2} - 2\zeta\omega_0 V\frac{\partial w}{\partial x'} + \omega_0^2 w = -g + p\delta(x') . \tag{4}$$

The boundary conditions are

$$\lim_{|x'|\to\infty} w = 0 . \tag{5}$$

To solve the problem of equation (4) and the boundary conditions (5), the Green's functions method is applied [6]. The Green's function can be built as a linear combination of the eigenfunctions of the differential operator of the equation (4). We start from the homogenous equation

$$(V^2 - c^2)\frac{\partial^2 w}{\partial x'^2} - 2\zeta\omega_0 V\frac{\partial w}{\partial x'} + \omega_0^2 w = 0 \tag{6}$$

and try the solution

$$w = Ae^{\lambda x'} . \tag{7}$$

The characteristic equation has the following form

$$(V^2 - c^2)\lambda^2 - 2\zeta\omega_0 V\lambda + \omega_0^2 = 0 \tag{8}$$

and its eigenvalues can be written as

$$\lambda_{1,2} = \omega_0 \frac{\zeta V \pm \sqrt{\zeta^2 V^2 + c^2 - V^2}}{V^2 - c^2} . \tag{9}$$

We have two cases, the so-called subcritical and overcritical cases.

1. The subcritical case – **$V < c$** – the force velocity is smaller than the velocity of the elastic wave in the contact wire. In this case, the eigenvalues are real and they have opposite signs

$$\lambda_1 = \omega_0 \frac{-\zeta V + \sqrt{\zeta^2 V^2 + c^2 - V^2}}{c^2 - V^2} > 0,$$

$$\lambda_2 = \omega_0 \frac{-\zeta V - \sqrt{\zeta^2 V^2 + c^2 - V^2}}{c^2 - V^2} < 0. \tag{10}$$

In fact, the Green's function $G(x',\xi)$ has two forms satisfying the boundary conditions

$$
\begin{aligned}
G^-(x',\xi) &= A^- e^{\lambda_1 x'} \quad \text{for} \quad -\infty < x' < \xi \\
G^+(x',\xi) &= A^+ e^{\lambda_2 x'} \quad \text{for} \quad \xi < x' < \infty,
\end{aligned}
\tag{11}
$$

where A^- and A^+ depend on the ξ variable. These functions will be calculated using both continuity and jump conditions [see ref. 6 and 7].

The Green's function has to be continuous in $x'=\xi$

$$
A^- e^{\lambda_1 \xi} = A^+ e^{\lambda_2 \xi}.
\tag{12}
$$

Its derivation in respect to x' has a jump in $x'=\xi$

$$
\frac{\partial G^+(\xi+0,\xi)}{\partial x'} - \frac{\partial G^-(\xi-0,\xi)}{\partial x'} = -\frac{1}{c^2 - V^2},
$$

respectively

$$
\lambda_2 A^+ e^{\lambda_2 \xi} - \lambda_1 A^- e^{\lambda_1 \xi} = -\frac{1}{c^2 - V^2}.
\tag{13}
$$

Upon solving the equations (12) and (13), it is obtained

$$
A^- = -\frac{e^{-\lambda_1 \xi}}{(\lambda_2 - \lambda_1)(c^2 - V^2)}, \quad A^+ = -\frac{e^{-\lambda_2 \xi}}{(\lambda_2 - \lambda_1)(c^2 - V^2)}
$$

and then the Green function

$$
\begin{aligned}
G^-(x',\xi) &= -\frac{e^{\lambda_1 (x'-\xi)}}{(\lambda_2 - \lambda_1)(c^2 - V^2)} \quad \text{for} \quad -\infty < x' < \xi \\
G^+(x',\xi) &= -\frac{e^{\lambda_2 (x'-\xi)}}{(\lambda_2 - \lambda_1)(c^2 - V^2)} \quad \text{for} \quad \xi < x' < \infty.
\end{aligned}
\tag{14}
$$

The Green function forms can be treated when considering that

$$
-(\lambda_2 - \lambda_1)(c^2 - V^2) = 2\omega_0 \sqrt{\zeta^2 V^2 + c^2 - V^2}.
\tag{15}
$$

If replacing (15) along with the values of λ_1 and λ_2 in (14), a 'condensed' form of the Green function derives

$$G(x',\xi) = \frac{e^{-\omega_0 \frac{\sqrt{\zeta^2 V^2 + c^2 - V^2}|x'-\xi| + \zeta V(x'-\xi)}{c^2 - V^2}}}{2\omega_0 \sqrt{\zeta^2 V^2 + c^2 - V^2}} . \qquad (16)$$

The wire travelling in the moving reference system is given by the solution of equation (3) written as

$$w(x') = \int_{-\infty}^{\infty} G(x',\xi)[-g + p\delta(\xi)]d\xi = -g\int_{-\infty}^{\infty} G(x',\xi)d\xi + pG(x',0) . \qquad (17)$$

In the end, the equation for the wire travelling is obtained

$$w(x') = -\frac{mg}{k} + \frac{p}{2\omega_0 \sqrt{\zeta^2 V^2 + c^2 - V^2}} \times \exp\left(-\omega_0 \frac{\sqrt{\zeta^2 V^2 + c^2 - V^2}|x'| + \zeta V x'}{c^2 - V^2} \right). \qquad (18)$$

The wire deformation in the point of application of the moving force will result if consider the above relation $x' = 0$. Here,

$$w(x' = 0) = -\frac{mg}{k} + \frac{p}{2\omega_0 \sqrt{\zeta^2 V^2 + c^2 - V^2}} .$$

The equation shows that the higher the wire deformation, the higher the speed, which is similar with the decrease in the dynamic rigidity of the contact wire.

2. The overcritical case ($c < V$) represents the overcritical case when force P travels at a higher speed than the wave propagation speed through the contact wire. Two different situations arise, namely $V < c/(1-\zeta^2)^{1/2}$ and $V > c/(1-\zeta^2)^{1/2}$.

Should $c < V < c/(1-\zeta^2)^{1/2}$, then the natural values are real and positive

$$\begin{aligned} \lambda_1 &= \omega_0 \frac{\zeta V - \sqrt{\zeta^2 V^2 - (V^2 - c^2)}}{V^2 - c^2} > 0, \\ \lambda_2 &= \omega_0 \frac{\zeta V + \sqrt{\zeta^2 V^2 - (V^2 - c^2)}}{V^2 - c^2} > 0 \end{aligned} \qquad (19)$$

and if $V > c/(1-\zeta^2)^{1/2}$, this will have the natural values be complex-conjugated with the real positive part

$$\begin{aligned} \lambda_1 &= \omega_0 \frac{\zeta V - j\sqrt{V^2 - \zeta^2 V^2 - c^2}}{V^2 - c^2} > 0, \\ \lambda_2 &= \omega_0 \frac{\zeta V + j\sqrt{V^2 - \zeta^2 V^2 - c^2}}{V^2 - c^2} > 0 \end{aligned} \qquad (20)$$

The only possible form of the Green function is

$$G^-(x',\xi) = A_1 e^{\lambda_1 x'} + A_2 e^{\lambda_2 x'} \quad \text{for} \quad -\infty < x' < \xi$$
$$G^+(x',\xi) = 0 \quad \text{for} \quad \xi < x' < \infty,$$

(21)

where A_1 and A_2 depend on ξ.

The continuity conditions of the function (12) and of the derivate jump (13) become

$$A_1 e^{\lambda_1 \xi} + A_2 e^{\lambda_2 \xi} = 0$$
$$\lambda_1 A_1 e^{\lambda_1 \xi} + \lambda_2 A_2 e^{\lambda_2 \xi} = -\frac{1}{V^2 - c^2}$$

(22)

A few calculations later, A_1 and A_2 write as

$$A_1 = -\frac{e^{-\lambda_1 \xi}}{(\lambda_1 - \lambda_2)(V^2 - c^2)}$$

$$A_2 = \frac{e^{-\lambda_2 \xi}}{(\lambda_1 - \lambda_2)(V^2 - c^2)}$$

(23)

as well as the Green function

$$G^-(x',\xi) = -\frac{e^{\lambda_1(x'-\xi)} - e^{\lambda_2(x'-\xi)}}{(\lambda_1 - \lambda_2)(V^2 - c^2)} \quad \text{for} \quad -\infty < x' < \xi$$
$$G^+(x',\xi) = 0 \quad \text{for} \quad \xi < x' < \infty.$$

(24)

The latter can be also written as

$$G(x',\xi) = -\frac{e^{\lambda_1(x'-\xi)} - e^{\lambda_2(x'-\xi)}}{(\lambda_1 - \lambda_2)(V^2 - c^2)} H(\xi - x'),$$

(25)

where $H(.)$ is Heaviside's unit step function.

The wire travelling reported to the moving reference system also uses the equation (20).

$$w(x') = -\frac{mg}{k} - \frac{p e^{\frac{\omega_0 \zeta V}{V^2 - c^2} x'}}{\omega_0 \sqrt{\zeta^2 V^2 - (V^2 - c^2)}} \times \text{sh}\left(\omega_0 \frac{\sqrt{\zeta^2 V^2 - (V^2 - c^2)}}{V^2 - c^2} x'\right) H(-x'). \quad (26)$$

3. NUMERICAL APPLICATION

The results of the numerical simulation are further presented by using the mechanical model above. The following data have been considered [1]: *m*=1.1 kg/m, *T*=15kN, *k*=0.4 kN/m^2, *a*=0.5 Ns/m^2 and *P*=55 N.

The values of the system parameters are as such

- natural frequency

$$v_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{400}{1,1}} = 3,03\,\text{Hz};$$

- damping degree

$$\zeta = \frac{0,5}{2\sqrt{400 \cdot 1,1}} = 0,0119\,;$$

-wave propagation speed through a wire

$$c = \sqrt{\frac{150000}{1,1}} = 116,8\;\text{m/s.}$$

- static deformation

$$w_0 = \frac{mg}{k} = \frac{1.1 \cdot 9.81}{400} = 0.027\,\text{m.}$$

Figure 3 a shows that, for the subcritical case (for simulation, *V*=70 m/s≈0.6*c*), the wire arrow falls exponentially on either side of the application point of the constant force. This decrease is more visible before the force rather after it. When damping is absent, the wire shape becomes symmetrical. Indeed, for *ζ*=0, equation (26) results as

$$w(x') = -\frac{mg}{k} + \frac{p}{2\omega_0\sqrt{c^2 - V^2}}\exp\left(-\omega_0|x'|\right),$$

which presents the symmetry of wire deformation compared to de *x′*=0.

For the overcritical case, *c*/(1-*ζ*^{2})$^{1/2}$ >*V* is the target because here the damping degree is very small and *c*/(1-*ζ*^{2})$^{1/2}$ is a little higher than *c*. In fact, *c*/(1-*ζ*^{2})$^{1/2}$ =1.000071*c*, which proves the previous statement.

Figure 3 b shows the wire deformation in an overcritical case. The calculation is for the speed of 130 m/s≈1.11*c*. While comparing the two figures 2 a and b, the difference between the subcritical and overcritical cases is visible.

For the overcritical case, the deformation is not symmetrical compared to the moving point of force application. Before the force, the wire is not deformed as the

force travels faster than the wave propagation speed through the wire. This perspective is similar with the Match effect in acoustics. On the contrary, there will be a wake zone after the force, where the wire deformation is very high and propagates on a significant distance, unlike the subcritical case where the wire deformation is located in the immediate vicinity of the force.
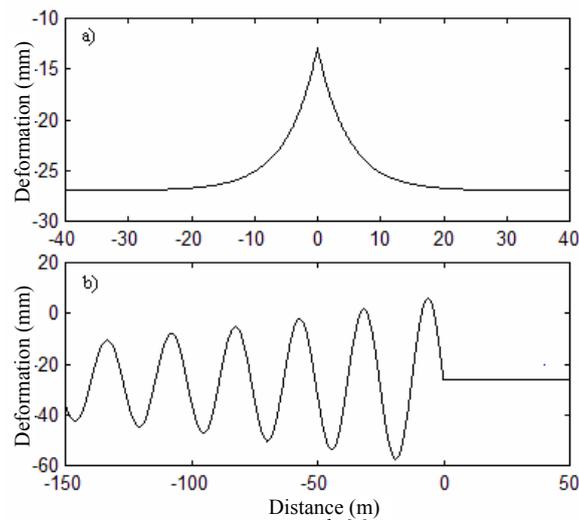

Fig. 3. Wire deformation:
a) subcritical speed; b) postcritical speed.

In the subcritical case, the maximum wire deformation is against the force, whereas the maximum deformation migrates somewhere after the force in the overcritical case.

4. CONCLUSIONS

The study of the catenary to a constant moving force is interesting from a practical point of view, thanks to the specific applications regarding the pantograph-catenary interaction for the high speed trains.

The paper focuses on the issues of catenary response to a moving force, starting from the reason that the contact wire can be assimilated with an infinite tensed chord connected to a fixed base by a viscous-elastic continuous support.

The solution of the problem is obtained by applying the Green function method. The results of the numerical simulation are consistent with those mentioned in the literature in review.

We have two cases, the sub-critical one, when the force velocity is lower than the critical velocity and the maximum wire deflection is against the contact point of force and the overcritical one, when the velocity of the force is higher than the critical velocity and the maximum wire deflection appears after the force position.

The later research will extend the application of this method to the distributed parameters model and periodical variation of the elasticity in the contact wire.

REFERENCES

[1]. **Metrikine, A.V.,** *Steady state response of an infinite string on a non-linear visco-elastic foundation to moving point loads*, Journal of Sound and Vibration 272 (2004), 1033-1046.

[2]. **Metrikine, A.V. şi Bosch, A.L.,** *Dynamic response of a two-level catenary to a moving load*, Journal of Sound and Vibration 292 (2006), 676-693.

[3]. **Mocică, G.,** *Transformarea Laplace şi operaţionale*, Ed. UPB, Bucureşti, 1993.

[4]. **Park, T.J., Han, C.S., Jang, J.H.,** *Dynamic sensivity analysis for the pantograph of a high-speed rail vehicle*, Journal of Sound and Vibration 266 (2003), 235-260.

[5]. **Sebeşan, I., Mazilu, T.,** *Vibraţiile vehiculelor feroviare* (Vibrations of railway vehicles), Editura MatrixRom, Bucureşti, 2010.

[6]. **Wu, T.X., Brennan, M.J.,** *Dynamic stiffness of a railway overhead wire system and its effect on pantograph-catenary system dynamics*, Journal of Sound and Vibration 219 (1999), 483-502.

[7]. **Wu, T.X., Brennan, M.J.,** *Basic Analytical Study of Pantograph-catenary System Dynamics*, Vehicle System Dynamics, 30 (1998), 443-456.